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Birefringence and dichroism of the QED vacuum

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Abstract. We use an analytic form for the Heisenberg–Euler Lagrangian to calculate the birefringent and dichroic properties of the vacuum for arbitrarily strong wrenchless fields.

1. Introduction

In the presence of a strong external field, the vacuum reacts, becoming magnetized and polarized. The index of refraction, magnetic permeability, and dielectric constant of the vacuum are straightforward to calculate using quantum electrodynamic one-loop corrections [1–6]. In this paper, we calculate the magnetic permeability and dielectric tensors of an external electric or magnetic field of arbitrary strength in terms of special functions. We combine these general results to calculate the complex-valued index of refraction as a function of field strength.

2. The permeability and polarizability of the vacuum

When one-loop corrections are included in the Lagrangian of the electromagnetic field one obtains a non-linear correction term:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1. \quad (1)$$

Both terms of the Lagrangian can be written in terms of the Lorentz invariants,

$$I = F_{\mu\nu} F^{\mu\nu} = 2(|\mathbf{B}|^2 - |\mathbf{E}|^2) \quad (2)$$

and

$$K = (\frac{1}{2}\epsilon^{\lambda\rho\mu\nu} F_{\lambda\rho} F_{\mu\nu})^2 = -(4\mathbf{E} \cdot \mathbf{B})^2 \quad (3)$$

following Heisenberg and Euler [7]. We use Greek indices to count over space and time components (0, 1, 2, 3) and Roman indices to count over spatial components only (1, 2, 3), and repeated indices imply summation.

Heisenberg and Euler [7] and Weisskopf [8] independently derived the effective Lagrangian of the electromagnetic field using electron-hole theory. Schwinger [9] later rederived the same result using quantum electrodynamics. In Heaviside–Lorentz units, the

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Lagrangian is given by

$$\mathcal{L}_0 = -\frac{1}{4}I \quad (4)$$

$$\mathcal{L}_1 = \frac{e^2}{hc} \int_0^\infty e^{-\zeta} \frac{d\zeta}{\zeta^3} \left\{ i\zeta^2 \frac{\sqrt{-K}}{4} \frac{\cos\left(\frac{\zeta}{B_k} \sqrt{-\frac{I}{2} + i\frac{\sqrt{-K}}{2}}\right) + \cos\left(\frac{\zeta}{B_k} \sqrt{-\frac{I}{2} - i\frac{\sqrt{-K}}{2}}\right)}{\cos\left(\frac{\zeta}{B_k} \sqrt{-\frac{I}{2} + i\frac{\sqrt{-K}}{2}}\right) - \cos\left(\frac{\zeta}{B_k} \sqrt{-\frac{I}{2} - i\frac{\sqrt{-K}}{2}}\right)} + |B_k|^2 + \frac{\zeta^2}{6}I \right\} \quad (5)$$

where $B_k = E_k = \frac{m^2 c^3}{e\hbar} \approx 1.3 \times 10^{16} \text{ V cm}^{-1} \approx 4.4 \times 10^{13} \text{ G}$.

In the weak-field limit Heisenberg and Euler [7] give

$$\mathcal{L} \approx -\frac{1}{4}I + E_k^2 \frac{e^2}{hc} \left[\frac{1}{E_k^4} \left(\frac{1}{180}I^2 - \frac{7}{720}K \right) + \frac{1}{E_k^6} \left(\frac{13}{5040}KI - \frac{1}{630}I^3 \right) \dots \right]. \quad (6)$$

We define a dimensionless parameter ξ to characterize the field strength

$$\xi = \frac{1}{E_k} \sqrt{\frac{I}{2}} \quad (7)$$

and use the analytic expansion of this Lagrangian for small K derived by Heyl and Hernquist [10]:

$$\mathcal{L}_1 = \mathcal{L}_1(I, 0) + K \left. \frac{\partial \mathcal{L}_1}{\partial K} \right|_{K=0} + \frac{K^2}{2} \left. \frac{\partial^2 \mathcal{L}_1}{\partial K^2} \right|_{K=0} + \dots \quad (8)$$

The first two terms of this expansion are given by

$$\mathcal{L}_1(I, 0) = \frac{e^2}{hc} \frac{I}{2} X_0\left(\frac{1}{\xi}\right) \quad (9)$$

$$\left. \frac{\partial \mathcal{L}_1}{\partial K} \right|_{K=0} = \frac{e^2}{hc} \frac{1}{16I} X_1\left(\frac{1}{\xi}\right) \quad (10)$$

where

$$X_0(x) = 4 \int_0^{x/2-1} \ln(\Gamma(v+1)) dv + \frac{1}{3} \ln\left(\frac{1}{x}\right) + 2 \ln 4\pi - \left(4 \ln A + \frac{5}{3} \ln 2\right) - \left[\ln 4\pi + 1 + \ln\left(\frac{1}{x}\right) \right] x + \left[\frac{3}{4} + \frac{1}{2} \ln\left(\frac{2}{x}\right) \right] x^2 \quad (11)$$

$$X_1(x) = -2X_0(x) + xX_0^{(1)}(x) + \frac{2}{3}X_0^{(2)}(x) - \frac{2}{9} \frac{1}{x^2} \quad (12)$$

and

$$X_0^{(n)}(x) = \frac{d^n X_0(x)}{dx^n} \quad (13)$$

$$\ln A = \frac{1}{12} - \zeta^{(1)}(-1) \approx 0.2488 \quad (14)$$

where $\zeta^{(1)}(x)$ denotes the first derivative of the Riemann zeta function.

We will treat the vacuum as a polarizable medium. In the Heaviside–Lorentz system, the macroscopic fields are given by the generalized momenta conjugate to the fields [5]

$$D = \frac{\partial \mathcal{L}}{\partial E} = E + P \quad H = -\frac{\partial \mathcal{L}}{\partial B} = B - M \quad P = \frac{\partial \mathcal{L}_1}{\partial E} \quad M = \frac{\partial \mathcal{L}_1}{\partial B}. \quad (15)$$

We define the vacuum dielectric and inverse magnetic permeability tensors as follows [11]

$$D_i = \epsilon_{ij} E_j \quad H_i = \mu'_{ij} B_j. \quad (16)$$

Using the definitions of I and K , we obtain

$$\epsilon_{ij} = \delta_{ij} - 4 \frac{\partial \mathcal{L}_1}{\partial I} \delta_{ij} - 32 \frac{\partial \mathcal{L}_1}{\partial K} B_i B_j \quad (17)$$

$$\mu'_{ij} = \delta_{ij} - 4 \frac{\partial \mathcal{L}_1}{\partial I} \delta_{ij} + 32 \frac{\partial \mathcal{L}_1}{\partial K} E_i E_j. \quad (18)$$

If we use the weak-field limit (equation (6)), we recover the results of Klein and Nigam [1]

$$\epsilon_{ij} = \delta_{ij} + \frac{1}{45\pi} \frac{\alpha}{B_k^2} [2(E^2 - B^2)\delta_{ij} + 7B_i B_j] \quad (19)$$

$$\mu'_{ij} = \delta_{ij} + \frac{1}{45\pi} \frac{\alpha}{B_k^2} [2(E^2 - B^2)\delta_{ij} - 7E_i E_j] \quad (20)$$

where the fine-structure constant α , equals $e^2/\hbar c$ in these units.

For wrenchless ($K = 0$) fields of arbitrary strength we use equations (11) and (12) to obtain

$$\epsilon_{ij} = \delta_{ij} + \frac{\alpha}{2\pi} \left\{ \left[-2X_0 \left(\frac{1}{\xi} \right) + \frac{1}{\xi} X_0^{(1)} \left(\frac{1}{\xi} \right) \right] \delta_{ij} - \frac{1}{\xi^2} \frac{B_i B_j}{B_k^2} X_1 \left(\frac{1}{\xi} \right) \right\} + \mathcal{O} \left[\left(\frac{\alpha}{2\pi} \right)^2 \right] \quad (21)$$

$$\mu'_{ij} = \delta_{ij} + \frac{\alpha}{2\pi} \left\{ \left[-2X_0 \left(\frac{1}{\xi} \right) + \frac{1}{\xi} X_0^{(1)} \left(\frac{1}{\xi} \right) \right] \delta_{ij} + \frac{1}{\xi^2} \frac{E_i E_j}{E_k^2} X_1 \left(\frac{1}{\xi} \right) \right\} + \mathcal{O} \left[\left(\frac{\alpha}{2\pi} \right)^2 \right]. \quad (22)$$

The expression for μ with only an external magnetic field agrees numerically with the results of Mielniczuk *et al* [6].

To examine wave propagation, we must first linearize the relations (equation (15)) in the fields of the wave ($\tilde{\mathbf{E}}, \tilde{\mathbf{B}}$) [4] and obtain a second set of matrices,

$$\tilde{\epsilon}_{ij} = \frac{\partial^2 \mathcal{L}}{\partial E_i \partial E_j} \quad (23)$$

$$\begin{aligned} &= \delta_{ij} - 4 \frac{\partial \mathcal{L}_1}{\partial I} \delta_{ij} + 16 \frac{\partial^2 \mathcal{L}_1}{\partial I^2} E_i E_j - \left(64K \frac{\partial^2 \mathcal{L}_1}{\partial K^2} + 32 \frac{\partial \mathcal{L}_1}{\partial K} \right) B_i B_j \\ &\quad + 128(\mathbf{E} \cdot \mathbf{B}) \frac{\partial^2 \mathcal{L}_1}{\partial I \partial K} (E_i B_j + E_j B_i) \end{aligned} \quad (24)$$

$$\tilde{\mu}'_{ij} = -\frac{\partial^2 \mathcal{L}}{\partial B_i \partial B_j} \quad (25)$$

$$\begin{aligned} &= \delta_{ij} - 4 \frac{\partial \mathcal{L}_1}{\partial I} \delta_{ij} - 16 \frac{\partial^2 \mathcal{L}_1}{\partial I^2} B_i B_j + \left(64K \frac{\partial^2 \mathcal{L}_1}{\partial K^2} + 32 \frac{\partial \mathcal{L}_1}{\partial K} \right) E_i E_j \\ &\quad + 128(\mathbf{E} \cdot \mathbf{B}) \frac{\partial^2 \mathcal{L}_1}{\partial I \partial K} (E_i B_j + E_j B_i). \end{aligned} \quad (26)$$

We use these matrices in the macroscopic Maxwell equations. To first order, $\tilde{\mathbf{H}} \parallel \tilde{\mathbf{B}}$ and $\tilde{\mathbf{D}} \parallel \tilde{\mathbf{E}}$, so we obtain the wave equation,

$$\nabla^2 \tilde{\mathbf{E}} - \frac{\tilde{\epsilon}}{\tilde{\mu}' c^2} \frac{\partial^2 \tilde{\mathbf{E}}}{\partial t^2} = 0 \quad (27)$$

and similarly for $\tilde{\mathbf{B}}$.

In equation (27), $\tilde{\mu}'$ and $\tilde{\epsilon}$ are the ratios of the macroscopic to the microscopic fields, i.e. $\tilde{\mathbf{H}} = \tilde{\mu}' \tilde{\mathbf{B}}$. The waves travel at a definite velocity $v = c\sqrt{\tilde{\mu}'/\tilde{\epsilon}}$ and the index of refraction is $n = \sqrt{\tilde{\epsilon}/\tilde{\mu}'}$.

If we take an external magnetic field parallel to the $\hat{3}$ axis, we obtain

$$\tilde{\epsilon}_{ij} = \delta_{ij} \left\{ 1 + \frac{\alpha}{2\pi} \left[-2X_0 \left(\frac{1}{\xi} \right) + \frac{1}{\xi} X_0^{(1)} \left(\frac{1}{\xi} \right) \right] \right\} - \delta_{i3}\delta_{j3} \frac{\alpha}{2\pi} X_1 \left(\frac{1}{\xi} \right) + \mathcal{O} \left[\left(\frac{\alpha}{2\pi} \right)^2 \right] \quad (28)$$

$$\begin{aligned} \tilde{\mu}'_{ij} = \delta_{ij} \left\{ 1 + \frac{\alpha}{2\pi} \left[-2X_0 \left(\frac{1}{\xi} \right) + \frac{1}{\xi} X_0^{(1)} \left(\frac{1}{\xi} \right) \right] \right\} \\ - \delta_{i3}\delta_{j3} \frac{\alpha}{2\pi} \left[X_0^{(2)} \left(\frac{1}{\xi} \right) \xi^{-2} - X_0^{(1)} \left(\frac{1}{\xi} \right) \xi^{-1} \right] + \mathcal{O} \left[\left(\frac{\alpha}{2\pi} \right)^2 \right]. \end{aligned} \quad (29)$$

In this case, we have the magnetic field of the wave either perpendicular to the plane containing the external magnetic field and the direction of propagation (\mathbf{k}), \perp mode, or in that plane, \parallel mode [5]. For the \perp mode, we obtain

$$n_{\perp} = 1 - \frac{\alpha}{4\pi} X_1 \left(\frac{1}{\xi} \right) \sin^2 \theta + \mathcal{O} \left[\left(\frac{\alpha}{2\pi} \right)^2 \right] \quad (30)$$

where θ is the angle between the direction of propagation and the external field. And for the \parallel mode, we obtain

$$n_{\parallel} = 1 + \frac{\alpha}{4\pi} \left[X_0^{(2)} \left(\frac{1}{\xi} \right) \xi^{-2} - X_0^{(1)} \left(\frac{1}{\xi} \right) \xi^{-1} \right] \sin^2 \theta + \mathcal{O} \left[\left(\frac{\alpha}{2\pi} \right)^2 \right]. \quad (31)$$

The expressions for n_{\parallel} , n_{\perp} obtained here are equivalent to those obtained by Tsai and Erber [12] through direct calculation of the photon propagator to one-loop accuracy in the presence of the external field.

If we take an external electric field parallel to the $\hat{3}$ axis, we obtain

$$\begin{aligned} \tilde{\epsilon}_{ij} = \delta_{ij} \left\{ 1 + \frac{\alpha}{2\pi} \left[-2X_0 \left(\frac{1}{\xi} \right) + \frac{1}{\xi} X_0^{(1)} \left(\frac{1}{\xi} \right) \right] \right\} \\ - \delta_{i3}\delta_{j3} \frac{\alpha}{2\pi} \left[X_0^{(2)} \left(\frac{1}{\xi} \right) \xi^{-2} - X_0^{(1)} \left(\frac{1}{\xi} \right) \xi^{-1} \right] + \mathcal{O} \left[\left(\frac{\alpha}{2\pi} \right)^2 \right] \end{aligned} \quad (32)$$

$$\tilde{\mu}'_{ij} = \delta_{ij} \left\{ 1 + \frac{\alpha}{2\pi} \left[-2X_0 \left(\frac{1}{\xi} \right) + \frac{1}{\xi} X_0^{(1)} \left(\frac{1}{\xi} \right) \right] \right\} - \delta_{i3}\delta_{j3} \frac{\alpha}{2\pi} X_1 \left(\frac{1}{\xi} \right) + \mathcal{O} \left[\left(\frac{\alpha}{2\pi} \right)^2 \right]. \quad (33)$$

In this case, the propagation modes have the electric field ($\tilde{\mathbf{E}}$) either in the \mathbf{k} - \mathbf{E} plane (\parallel mode) or perpendicular to the plane. For an external electric field, we define

$$\xi = iy = i \frac{E}{E_k} \quad (34)$$

and substitute this into equations (30) and (31). This yields indices of refraction

$$n_{\perp} = 1 + \frac{\alpha}{4\pi} X_1 \left(\frac{1}{iy} \right) \sin^2 \theta + \mathcal{O} \left[\left(\frac{\alpha}{2\pi} \right)^2 \right] \quad (35)$$

$$n_{\parallel} = 1 + \frac{\alpha}{4\pi} \left[X_0^{(2)} \left(-\frac{i}{y} \right) y^{-2} - i X_0^{(1)} \left(-\frac{i}{y} \right) y^{-1} \right] \sin^2 \theta + \mathcal{O} \left[\left(\frac{\alpha}{2\pi} \right)^2 \right] \quad (36)$$

where θ again refers to the angle between the direction of propagation and the external electric field.

In the weak-field limit, we have [10]

$$X_1\left(\frac{1}{\xi}\right) = -\frac{14}{45}\xi^2 + \mathcal{O}(\xi^4) \tag{37}$$

$$X_0^{(2)}\left(\frac{1}{\xi}\right)\xi^{-2} - X_0^{(1)}\left(\frac{1}{\xi}\right)\xi^{-1} = \frac{8}{45}\xi^2 + \mathcal{O}(\xi^4). \tag{38}$$

An external electric field gives $\xi^2 < 0$ and an external magnetic field gives $\xi^2 > 0$, therefore $n_{\perp}, n_{\parallel} > 1$ in the weak-field limit for both cases. Using this limit in equations (31) and (30) yields weak-field expressions for the index of refraction in a magnetic field in agreement with earlier work [4, 5].

2.1. Series and asymptotic expressions

To calculate the indices of refraction in the weak- and strong-field limit, we use the expansions of Heyl and Hernquist [10]. For an external magnetic field, in the weak-field limit ($\xi < 0.5$),

$$n_{\perp} = 1 + \frac{\alpha}{4\pi} \sin^2 \theta \left[\frac{14}{45}\xi^2 - \frac{1}{3} \sum_{j=2}^{\infty} \frac{2^{2j}(6B_{2(j+1)} - (2j+1)B_{2j})}{j(2j+1)} \xi^{2j} \right] + \mathcal{O}\left[\left(\frac{\alpha}{2\pi}\right)^2\right] \tag{39}$$

$$n_{\parallel} = 1 - \frac{\alpha}{4\pi} \sin^2 \theta \sum_{j=1}^{\infty} \frac{2^{2(j+1)}B_{2(j+1)}}{2j+1} \xi^{2j} + \mathcal{O}\left[\left(\frac{\alpha}{2\pi}\right)^2\right]. \tag{40}$$

In the strong-field limit ($\xi > 0.5$), we obtain

$$\begin{aligned} n_{\perp} = 1 + \frac{\alpha}{4\pi} \sin^2 \theta & \left[\frac{2}{3}\xi - \left(8 \ln A - \frac{1}{3} - \frac{2}{3}\gamma\right) - \left(\ln \pi + \frac{1}{18}\pi^2 - 2 - \ln \xi\right) \xi^{-1} \right. \\ & \left. - \left(-\frac{1}{2} - \frac{1}{6}\zeta(3)\right) \xi^{-2} \right. \\ & \left. - \sum_{j=3}^{\infty} \frac{(-1)^{j-1}}{2^{j-2}} \left(\frac{j-2}{j(j-1)}\zeta(j-1) + \frac{1}{6}\zeta(j+1)\right) \xi^{-j} \right] + \mathcal{O}\left[\left(\frac{\alpha}{2\pi}\right)^2\right] \end{aligned} \tag{41}$$

$$\begin{aligned} n_{\parallel} = 1 + \frac{\alpha}{4\pi} \sin^2 \theta & \left[\frac{2}{3} - \frac{\ln \xi + 1 - \ln \pi}{\xi} - \frac{1}{\xi^2} + \sum_{j=3}^{\infty} \frac{(-1)^{j-1}}{2^{j-2}} \frac{j-2}{j-1} \zeta(j-1) \xi^{-j} \right] \\ & + \mathcal{O}\left[\left(\frac{\alpha}{2\pi}\right)^2\right] \end{aligned} \tag{42}$$

where γ is Euler's constant.

For an external electric field, in the weak-field limit ($y < 0.5$) we obtain,

$$n_{\perp} = 1 + \frac{\alpha}{4\pi} \left[\frac{14}{45}y^2 + \frac{1}{3} \sum_{j=2}^{\infty} \frac{(-1)^j 2^{2j}(6B_{2(j+1)} - (2j+1)B_{2j})}{j(2j+1)} y^{2j} \right] + \mathcal{O}\left[\left(\frac{\alpha}{2\pi}\right)^2\right] \tag{43}$$

$$n_{\parallel} = 1 + \frac{\alpha}{4\pi} \sin^2 \theta \sum_{j=1}^{\infty} \frac{(-1)^j 2^{2(j+1)}B_{2(j+1)}}{2j+1} y^{2j} + \mathcal{O}\left[\left(\frac{\alpha}{2\pi}\right)^2\right] \tag{44}$$

and in the strong-field limit ($y > 0.5$)

$$n_{\perp} = 1 + \frac{\alpha}{4\pi} \sin^2 \theta \left[-i \frac{2}{3} y + \left(8 \ln A - \frac{1}{3} - \frac{2}{3} \gamma \right) - i \left(\ln \pi + \frac{1}{18} \pi^2 - 2 - \ln(iy) \right) y^{-1} \right. \\ \left. - \left(-\frac{1}{2} - \frac{1}{6} \zeta(3) \right) y^{-2} \right. \\ \left. + \sum_{j=3}^{\infty} \frac{(-1)^{j-1}}{2^{j-2}} \left(\frac{j-2}{j(j-1)} \zeta(j-1) + \frac{1}{6} \zeta(j+1) \right) (iy)^{-j} \right] + \mathcal{O} \left[\left(\frac{\alpha}{2\pi} \right)^2 \right] \quad (45)$$

$$n_{\parallel} = 1 - \frac{\alpha}{4\pi} \sin^2 \theta \left[\frac{2}{3} + i \frac{\ln(iy) + 1 - \ln \pi}{y} + \frac{1}{y^2} + \sum_{j=3}^{\infty} \frac{(-1)^{j-1}}{2^{j-2}} \frac{j-2}{j-1} \zeta(j-1) (iy)^{-j} \right] \\ + \mathcal{O} \left[\left(\frac{\alpha}{2\pi} \right)^2 \right]. \quad (46)$$

From this equation, it is apparent that the index of refraction acquires an imaginary part in strong electric fields.

3. Birefringence

In general, the birefringence is quantified by the difference of the indices of refraction for the two modes of propagation,

$$n_{\perp} - n_{\parallel} = \pm \frac{\alpha}{4\pi} \left[X_0^{(1)} \left(\frac{1}{\xi} \right) \xi^{-1} - X_0^{(2)} \left(\frac{1}{\xi} \right) \xi^{-2} - X_1 \left(\frac{1}{\xi} \right) \right] \sin^2 \theta + \mathcal{O} \left[\left(\frac{\alpha}{2\pi} \right)^2 \right] \quad (47)$$

where the upper sign is for the magnetized case and the lower sign for the electrified case. Figure 1 depicts the indices of refraction for these two cases.

4. Dichroism

The analytic properties of the function $n_{\parallel}(\xi)$ can be used to estimate the dichroic properties of a magnetized or electrified vacuum. In an external electric field we have $\xi = iE/E_k = iy$, while in a magnetic field $\xi = B/B_k$. $n_{\parallel}(\xi)$ is real for real arguments; however, for imaginary

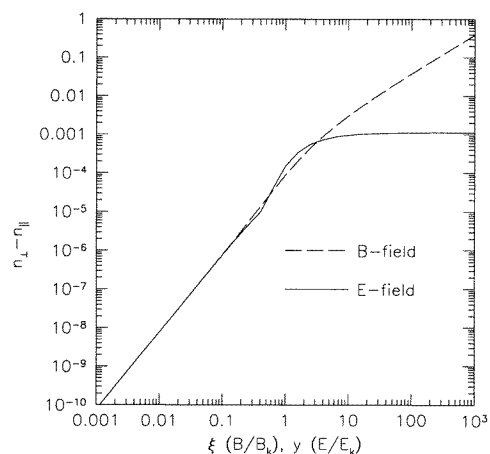


Figure 1. The difference between the index of refraction of the parallel and perpendicular polarizations for light travelling through external electric or magnetic fields.

ξ , $n_{\parallel}(\xi)$ acquires an imaginary part. Classically, this imaginary part may be related to the attenuation length of a plane wave traversing the vacuum

$$l = \frac{2\pi\lambda}{\text{Im } n} \tag{48}$$

where λ is the wavelength of the radiation. In quantum-field theory, the imaginary part of n is related to the imaginary part of the photon polarization operator and therefore the cross section for one-photon pair production.

In general the imaginary part for the two polarization modes is

$$\text{Im } n_{\perp} = \frac{\alpha}{4\pi} \sin^2 \theta \text{Im } X_1 \left(-\frac{i}{y} \right) + \mathcal{O} \left[\left(\frac{\alpha}{2\pi} \right)^2 \right] \tag{49}$$

$$\text{Im } n_{\parallel} = \frac{\alpha}{4\pi} \sin^2 \theta \left[\text{Im } X_0^{(2)} \left(-\frac{i}{y} \right) y^{-2} - \text{Re } X_0^{(1)} \left(-\frac{i}{y} \right) y^{-1} \right] + \mathcal{O} \left[\left(\frac{\alpha}{2\pi} \right)^2 \right]. \tag{50}$$

These are conveniently calculated by evaluating the imaginary part of $X_0(x)$ for imaginary values of x by integrating around the poles of equation (6) [10, 13],

$$\text{Im } X_0(x) = -\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{e^{-i\pi nx}}{n^2} = -\frac{1}{\pi} e^{-\pi/y} {}_{1,1,1}F_{2,2}(e^{-\pi/y}) \tag{51}$$

where F is a generalized hypergeometric function. Using equation (12) to calculate $\text{Im } X_1(x)$, yielding for the indices of refraction,

$$\text{Im } n_{\perp} = \frac{\alpha}{4\pi} \sin^2 \theta \sum_{n=1}^{\infty} \left(\frac{2}{3}\pi + \frac{1}{n} \frac{1}{y} + \frac{1}{n^2} \frac{2}{\pi} \right) e^{-n\pi/y} + \mathcal{O} \left[\left(\frac{\alpha}{2\pi} \right)^2 \right] \tag{52}$$

$$= \frac{\alpha}{4\pi} \sin^2 \theta \left[\frac{2}{3}\pi (e^{\pi/y} - 1)^{-1} - \frac{1}{y} \ln(1 - e^{-\pi/y}) + \frac{2}{\pi} e^{-\pi/y} {}_{1,1,1}F_{2,2}(e^{-\pi/y}) \right] + \mathcal{O} \left[\left(\frac{\alpha}{2\pi} \right)^2 \right] \tag{53}$$

$$\text{Im } n_{\parallel} = \frac{\alpha}{4\pi} \sin^2 \theta \sum_{n=1}^{\infty} \left(\frac{\pi}{y^2} + \frac{1}{n} \frac{1}{y} \right) e^{-n\pi/y} + \mathcal{O} \left[\left(\frac{\alpha}{2\pi} \right)^2 \right] \tag{54}$$

$$= \frac{\alpha}{4\pi} \sin^2 \theta \left[\frac{\pi}{y^2} (e^{\pi/y} - 1)^{-1} - \frac{1}{y} \ln(1 - e^{-\pi/y}) \right] + \mathcal{O} \left[\left(\frac{\alpha}{2\pi} \right)^2 \right]. \tag{55}$$

Figure 2 depicts the imaginary part of the index of refraction as a function of field strength.

In the weak-field limit, the imaginary part of the index of refraction is exponentially small as Klein and Nigam [2] found. However, our result is larger by a factor of $1/y$ in this limit and is more complicated. The error occurs between their equations (5) and (6). First, they have neglected the real part of the integral, and as in [1], they have calculated μ'_{ij} and used it as μ_{ij} . These errors are not important for this application. However, their function $\Phi_2(x)$ has not been calculated correctly. By examination of their equation (6), we see that

$$\frac{\partial \mathcal{L}}{\partial \mathcal{K}} = -i \frac{\alpha}{2} \frac{1}{16l} \Phi_2(x) \tag{56}$$

so

$$\Phi_2(x) = -\frac{1}{\pi} \text{Im } X_1 \left(\frac{\pi}{ix} \right) \tag{57}$$

which from examination of equation (53) is significantly more complicated than their expression.

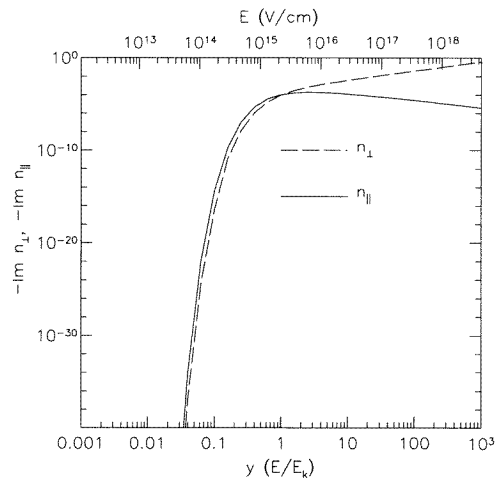


Figure 2. The imaginary part of the index of refraction for perpendicular and parallel propagation modes for light travelling through an external electric field.

5. Conclusions

Using a closed-form expression for the Heisenberg–Euler effective Lagrangian for quantum electrodynamics in wrenchless ($K = 0$) fields, we have calculated general expressions for the index of refraction of a slowly varying electromagnetic field, and evaluated these expressions for the simple cases of a pure electric or magnetic field. Our results agree with some previous work [4–6, 12] in the appropriate limits. We expect these results to be of general utility especially in the study of light propagation in the vicinity of strongly magnetized neutron stars.

Acknowledgments

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